

AFFINE-COMPACT FUNCTORS

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ABSTRACT. Several well known polytopal constructions are examined from the functorial point of view. A naive analogy between the Billera-Sturmfels fiber polytope and the abelian kernel is disproved by an infinite explicit series of polytopes. A correct functorial formula is provided in terms of the affine-compact kernel. The dual cokernel object is almost always the natural affine projection. The Mond-Smith-van Straten space of sandwiched simplices, useful in stochastic factorizations, leads to a different kind of affine-compact functors and new challenges in polytope theory.

1. INTRODUCTION

The role of representable functors in algebraic geometry and topology is well known. In this work we initiate a functorial approach to various convex and polytopal constructions, with similar emphasis on representable functors.

On the one hand, the category **Pol** of *convex polytopes* and *affine maps* is ‘too linear’ for the two mentioned disciplines. But, on the other hand, the rich combinatorial structure it carries makes **Pol** the backbone of geometric and, to a large extent, algebraic combinatorics. Methods and techniques from algebraic topology and geometry are often used to solve important open problems in combinatorics. A natural question is whether **Pol** itself can be subjected to a categorical/homological analysis. Put another way, one can ask (i) whether the representable functors of well-known polytopal objects have expected properties within the category of functors defined on **Pol**, and (ii) whether natural polytopal correspondences are representable functors, leading to new geometric objects. An indication that these are meaningful questions is the initial observation that **Pol** is a self-enriched symmetric monoidal category. That $\text{Hom}(P, Q)$ and $P \otimes Q$ are also polytopes for any $P, Q \in \mathbf{Pol}$, with the readily described facets of $\text{Hom}(P, Q)$ and vertices of $P \otimes Q$, is an easy exercise. However, determination of the vertices of $\text{Hom}(P, Q)$ and facets of $P \otimes Q$ is a real challenge and partial progress in this direction is accomplished in [5, 8].

This work represents the next natural step beyond Hom and \otimes . Namely,

- In Section 4 we examine the Billera-Sturmfels *fiber polytope* Σf [3] from the functorial perspective. This polytope is the average over the fibers of a map f in **Pol** and defined in terms of the Minkowski integral. Fiber polytopes have many applications, most admittedly in triangulation theory [7, Ch.9]. They are reminiscent of

2010 *Mathematics Subject Classification.* Primary 18B30, 52B11; Secondary 52A07, 52A25.

Key words and phrases. polytope, fiber polytope, compact set, convex set, affine map, Minkowski sum, representable functor, affine-compact kernel, sandwiched simplices.

Supported by U.S. NSF grant DMS 1301487 and Georgian NSF grant DI/16/5-103/12.

the kernels of linear maps, but only informally as the category **Pol** is far from being abelian – it even lacks 0 object. Still, one can ask whether for R and $f : P \rightarrow Q$ in **Pol** the polytopes $\text{Hom}(R, \Sigma f)$ and $\Sigma \text{Hom}(R, f)$ are isomorphic, mimicking the functorial isomorphism $\text{Hom}(-, \ker \alpha) \cong \ker \text{Hom}(-, \alpha)$ in the abelian setting. In Theorem 4.1 we provide an infinite series of polytopal counterexamples.

- In Section 6 we develop an affine-compact version of the linear kernel for the more general category **Comp** of all *convex compact sets* and affine maps. It leads to the correct version (Theorem 6.2) of the naive fiber equality, which was disproved in Section 4. The *affine-compact kernel* is preceded in Section 3 by a similar analysis of the Minkowski sum. Even if one wants to work exclusively with **Pol**, the limit sets enter the picture via the proof of the central Lemma 5.2. This makes the passage from **Pol** to **Comp** even more natural.

- In Section 7 we show that the dual concept of the affine-compact cokernel is less geometrically meaningful: for a map $f : X \rightarrow Y$ in **Comp**, the set $\text{coker}_{\text{Aff}}(f)$ is (almost always) the linear projection of Y along the affine hull of $f(X)$.

- Section 8 represents a more radical departure from the linear setup. Motivated by the *space of sandwiched simplices*, which was introduced by Mond-Smith-van Straten [11] for modeling stochastic factorizations, we define a pair of functors: *sandwiching* and *complementing*. Informally, for a map $\varphi : Y \rightarrow X$ one functor makes $\varphi(Y)$ a necessary target and the other makes the interior of $\varphi(Y)$ an impossible target. We observe that these functors are still *affine-compact* but with values considerably beyond **Comp**; unlike the sandwiches, the topological behavior of the complementing functor is transparent, and there is a complementarity between the functors (Theorem 8.2). Expected functorial properties lead to interesting questions on polytopes.

Acknowledgment. Thanks to Tristram Bogart from whom I learned about the question (attributed to someone else) whether $\Sigma \text{Hom}(R, f) \cong \text{Hom}(R, \Sigma f)$.

2. PRELIMINARIES

We refer the reader to (i) the classics [10] for the standard material on categories and [9] for the enriched context, and (ii) [6, Ch.1] and [14] for basic facts on polytopes. In particular, our polytopes are always assumed to be *convex*.

Following [10], for categories \mathcal{C} and \mathcal{D} , we denote by \mathcal{C}^{op} the dual category of \mathcal{C} and by $\mathcal{C}^{\mathcal{D}}$ the category of covariant functors $\mathcal{D} \rightarrow \mathcal{C}$ and natural transformations. Our category of vector spaces **Vect** consists of the subspaces of \mathbb{R}^n , $n \geq 0$, and linear maps. Every $V \in \mathbf{Vect}$ comes equipped with the Euclidean norm and d -volume, $d = \dim V$. A convex set is always assumed to be in an ambient space $V \in \mathbf{Vect}$. By **Pol** and **Comp** we denote the categories of polytopes and compact convex sets and their *affine maps* (i.e., those respecting barycentric coordinates), respectively. Affine maps make sense for arbitrary, not necessarily convex, subsets of vector spaces. Under an *affine-compact functor* we mean a functor, defined on **Pol** or **Comp**, evaluating in the category of general compact subsets of vector spaces,

and inducing affine maps between the hom-sets. By $\text{Hom}(-, -)$ we always mean affine maps. For the linear maps we will use $\text{Hom}_{\mathbf{Vect}}(-, -)$.

That **Pol** is a self-enriched symmetric monoidal category is an easy observation; a sketch is given in [5]. The tensor product in **Pol** is the *dehomogenization* of the tensor product of the associated *homogenization cones*. The work [5] focuses on the geometric structure of the hom-polytopes $\text{Hom}(P, Q)$. More background material on Hom and \otimes in the category of general convex cones is found [13], which focuses on multilinear optimization. The undergraduate thesis [12] makes a lucid reading on categorial generalities on polytopes and cones. The category **Comp** extends the self-enriched symmetric monoidal structure from **Pol** via $X \otimes X' := \bigcup(P \otimes Q \mid P \subset X \text{ and } Q \subset X' \text{ polytopes})$. There is another self-enriched symmetric monoidal extension of **Pol**, based on *polytopal complexes* [1].

For a subset X of a vector space, the *convex* and *affine hulls* will be denoted, correspondingly, by $\text{conv}(X)$ and $\text{Aff}(X)$. For a convex set X , by $\text{int}(X)$ we denote the *relative interior* of X . The *boundary* of X is $\partial X = X \setminus \text{int}(X)$. For a polytope P the set of its *vertices*, that of *facets*, and the *normal fan* will be denoted by $\text{vert}(P)$, $\mathbb{F}(P)$, and $\mathcal{N}(P)$, respectively. All further terminology and notation will be introduced in the text.

3. MINKOWSKI SUMS, FIBERS, CONTINUITY

Let $V \in \mathbf{Vect}$, $\lambda_1, \dots, \lambda_n \in \mathbb{R}_{\geq 0}$, and $X_1, \dots, X_n \subset V$ be subsets. The corresponding *Minkowski linear combination* is $\sum_{i=1}^n \lambda_i X_i = \left\{ \sum_{i=1}^n \lambda_i x_i \mid x_i \in X_i, i = 1, \dots, n \right\}$. It belongs to **Comp** if $X_i \in \mathbf{Comp}$, or to **Pol** if $X_i \in \mathbf{Pol}$.

Lemma 3.1. *Let P and Q be polytopes in a same vector space.*

- (a) $\mathcal{N}(P + Q)$ is the common refinement of $\mathcal{N}(P)$ and $\mathcal{N}(Q)$.
- (b) $\#\text{vert}(P) \leq \#\text{vert}(P + Q)$ and $\#\mathbb{F}(P) \leq \#\mathbb{F}(P + Q)$.
- (c) If $\mathcal{N}(P) = \mathcal{N}(Q)$ then the faces of $P + Q$ are the Minkowski sums of the pairs of corresponding faces of P and Q .

The part (a) is proved in [14, Proposition 7.12], and (b,c) are easy consequences.

Definition 3.2. Let $f : X \rightarrow Y$ be a map in **Comp**. The set of *sections* of f is defined by $\Gamma_f = \{\gamma : f(X) \rightarrow X \mid \gamma \text{ is measurable and } f \circ \gamma = \mathbf{1}_{f(X)}\}$. The *fiber* of f is defined by

$$\Sigma f = \frac{1}{\text{vol}(f(X))} \left\{ \int_Y \gamma(y) \mid \gamma \in \Gamma_f \right\} \subset V.$$

Remark 3.3. Informally, Σf is the ‘average fiber’ of f over $f(X)$. Definition 3.2 is the straightforward extension of the Billera-Sturmfels *fiber polytope* [3] to the class of compact convex sets. Unlike [3], we allow non-surjective maps f because even if $f : X \rightarrow Y$ is surjective the induced map $\text{Hom}(Z, f)$, important in our analysis of Σ , may fail to be surjective; e.g., for a surjective affine map from a tetrahedron X to a quadrangle $Y = Z$ the identity map $\mathbf{1}_Z$ does not lift to $\text{Hom}(Z, X)$.

Proposition 3.4. *Let $f : X \rightarrow Y$ be in **Comp**.*

(a) If X and Y are in **Pol** then Σf is the following polytope

$$\Sigma f = \sum_{i=1}^n \frac{\text{vol}(\sigma_i)}{\text{vol}(f(P))} \cdot f^{-1}(x_i),$$

where $\{\sigma_1, \dots, \sigma_n\}$ are the maximal cells of any subdivision of $f(X)$, subdividing the f -images of the faces of the polytope X , and $x_i \in \sigma_i$ are the barycenters.

(b) $\Sigma f \in \mathbf{Comp}$ and $\dim(\Sigma f) = \dim(X) - \dim(f(X)) =: \text{codim } f$.

Proof. (a) This is a slightly extended reformulation of [3, Theorem 1.5], which states the equality for the coarsest such subdivision. But exactly the same argument applies to any subdivision. Alternatively, the general case reduces to the case when the coarsest subdivision is the trivial subdivision of Y , and then the claim follows from the fact that $y \mapsto f^{-1}(y)$ respects affine combinations.

(b) The argument in the polytopal case [3, Proposition 1.1], using Aumann's 1965 results on integrals of set-valued functions, works here too. Alternatively, the general case can be deduced from the polytopal case by approximating X from outside by a nested set of polytopes in $\text{Aff}(X)$, containing X , and approximating Y from outside by the images of these polytopes under the affine extension $\tilde{f} : \text{Aff}(X) \rightarrow \text{Aff}(Y)$. In this case Σf is the intersection of the resulting fiber polytopes.

The dimension equality is a consequence of (a). \square

For a subset X of a vector space V and a real number $\delta > 0$, the δ -neighborhood of X will be denoted by $U_\delta(X)$; i.e., $U_\delta(X)$ is the union of open δ -discs, centered at the elements of X .

Definition 3.5. Let $\varepsilon > 0$ and $V \in \mathbf{Vect}$. Assume $X_t, X \in \mathbf{Comp}$ and $X_t, X \subset V$ for $0 < t < \varepsilon$. We write $\lim_{t \rightarrow 0} X_t = X$ if for every $\delta > 0$ the inclusions $X_t \subset U_\delta(X)$ and $X \subset U_\delta(X_t)$ are satisfied for all sufficiently small t .

The limit, if it exists, is unique. The uniqueness would fail if we allowed non-closed convex sets.

Lemma 3.6. Let $V, W \in \mathbf{Vect}$, $X_t, Y_t, X, Y \in \mathbf{Comp}$, $X_t, X \subset V$, and $Y_t, Y \subset W$ for $0 < t < \varepsilon$. Assume $f : X \rightarrow Y$ is an affine map. If $\lim_{t \rightarrow 0} X_t = X$ and $\lim_{t \rightarrow 0} Y_t = Y$ then:

- (a) $\lim_{t \rightarrow 0} (X_t + Y_t) = X + Y$, assuming $V = W$;
- (b) $\lim_{t \rightarrow 0} \text{Hom}(X_t, Y_t) = \text{Hom}(X, Y)$ in $\text{Hom}(\text{Aff}(X), W)$, assuming $\text{Aff}(X) = \text{Aff}(X_t)$ for all t ;
- (c) $\lim_{t \rightarrow 0} \Sigma f_t = \Sigma f$, assuming $\text{Aff}(X) = \text{Aff}(X_t)$ for all t , where $f_t : X_t \rightarrow Y_t$ is obtained from f by first extending to $\text{Aff}(X)$ and then restricting to X_t .

Proof. (a) is obvious in view of the uniqueness of a limit.

In (b) the condition on affine hulls is needed for an ambient vector space, where the convergence occurs. (We think of $\text{Hom}(\text{Aff}(X), W)$ as $W^{\dim X + 1}$.) Without loss of generality we can assume $\text{Aff}(X) = V$ and that there exists an affinely independent set $\{x_0, \dots, x_d\} \subset X \cap (\bigcap_{0 < t < \varepsilon} X_t)$, $d = \dim V$. Then there are infinitesimally small perturbations of the images $f(x_i)$ as $t \rightarrow 0$ such that the perturbed maps $f_t \in$

$\text{Hom}(V, W)$ first bring $f_t(V)$ into $\text{Aff}(Y_t)$ and then ensure the inclusions $f_t(X_t) \subset Y_t$. This implies the inclusion \supset , and the other inclusion is more straightforward.

(c) This is an easy exercise on integrals. \square

4. NO Σ -COVARIANCE

Let P, Q, R and $f : P \rightarrow Q$ be in **Comp**. The convex sets $\Sigma \text{Hom}(R, f)$ and $\text{Hom}(R, \Sigma f)$ are not completely unrelated: (i) if P, Q, R are polytopes then so are these sets, (ii) both have dimension $(\dim R + 1) \text{codim } f$ (follows from Proposition 3.4(b)), and (iii) $\Sigma \text{Hom}(R, f) \cong \text{Hom}(R, \Sigma f)$ in either of the following three cases: $P = P' \times Q$ and f is the projection map, Q is a point, f is injective. But the similarities end here.

The following formula for a centrally symmetric d -polytope $S \subset \mathbb{R}^d$ with respect to 0 is contained in [5, Corollary 3.6]:

$$(1) \quad \text{Hom}(S, [0, 1]) \cong \diamond(S^\circ)$$

where S° is the *polar* of S and, for any polytope $T \subset \mathbb{R}^d$ with $0 \in \text{int}(T)$, $\diamond(T)$ is the *bipyramid* $\text{conv}((T, 0), (0, 1), (0, -1)) \subset \mathbb{R}^{d+1}$.

Theorem 4.1. *Let $Q \subset \mathbb{R}^2$ be a centrally symmetric polygon. Assume $P \subset \mathbb{R}^3$ is a polytope, such that $Q \times [0, \varepsilon] \subset P$ for some $\varepsilon > 0$, P is combinatorially equivalent to a prism over Q , and the opposite facet $Q' \subset P$ is not parallel to Q ; i.e., P is a truncated right prism over Q . Then $\# \text{vert}(\Sigma \text{Hom}(R, f)) \geq \# \text{vert}(\text{Hom}(R, \Sigma f)) + 2$ for the orthogonal projection $f : P \rightarrow Q$. In particular, $\Sigma \text{Hom}(R, f) \not\cong \text{Hom}(R, \Sigma f)$.*

Proof. Denote $f_* = \text{Hom}(R, f)$. Let Q be an $2n$ -gon and $\{v_1, \dots, v_{2n}\} = \text{vert}(Q)$, the indexing being cyclic and mod $(2n)$. The polar Q° is also a centrally symmetric $2n$ -gon. We will identify Q with $(Q, 0)$.

For any map $\alpha \in \text{Hom}(Q, Q)$ the preimage $f_*^{-1}(\alpha) \subset \text{Hom}(Q, P)$ is a subpolytope, generically of dimension 3. Next we prove the implication

$$(2) \quad \text{rank } \alpha = 2 \implies \# \text{vert}(f_*^{-1}(\alpha)) \geq 2n + 4.$$

Assume $\text{rank } \alpha = 2$. Then the subpolytope $Q_\alpha := \alpha(Q) \subset Q$ is isomorphic to Q . Let P_α be the maximal truncated right prism inside P with Q_α as the base. Denote by Q'_α the facet of P_α , opposite to Q_α . Let $w_i = \alpha(v_i)$ and w'_i be the corresponding vertices of Q'_α .

The elements of $f_*^{-1}(\alpha)$ can be interpreted as the affine planes in $H \subset \mathbb{R}^3$, meeting all vertical edges of P_α : if $\{x_i\} = (\alpha(v_i) \times \mathbb{R}_{\geq 0}) \cap H$ then the map corresponding to H is defined by $v_i \mapsto x_i$, $i = 1, \dots, 2n$. After this interpretation, the vertices of $f_*^{-1}(\alpha)$ correspond to the planes H which do not fit in a *smooth 1-family* of affine planes, satisfying the same condition. Here, under a ‘smooth 1-family’ we mean a system $\{H_t\}_{(-1,1)}$ of affine planes in \mathbb{R}^3 such that the intersection point $H_t \cap (\mathbb{R}_{\geq 0}(0, 0, 1))$ and the unit normals to H_t are both smooth functions of t , and we say that H ‘fits’ in such a system if $H = H_0$. This *smooth perturbation criterion* for the vertices of a polytope is crucial in [5, 8] for studying the vertex sets of various hom-polytopes. The planes H , corresponding to the vertices of $f_*^{-1}(\alpha)$, will be called *tight*.

For every index i , we can rotate the coordinate plane $(\mathbb{R}^2, 0)$ in \mathbb{R}^3 about the axis $\text{Aff}(w_i, w_{i+1})$, staying within the family of planes corresponding to $f_*^{-1}(\alpha)$, until we hit the polygon Q'_α . Let H_i be the corresponding extremal position of the rotated plane. Then H_i is tight, representing a vertex $z_i \in f_*^{-1}(\alpha)$. Similarly, every edge $[w'_i, w'_{i+1}] \subset Q'_\alpha$ gives rise to a vertex $z'_i = f_*^{-1}(\alpha)$. We have $z_i \neq z_j$ and $z'_i \neq z'_j$ for $i \neq j$, and $z_i = z'_j$ if the plane $H_i = H'_j$ contains the corresponding edges of Q_α and Q'_α . In particular, if there is an index i such that $H_i \cap Q'_\alpha \in \text{vert}(Q'_\alpha)$ then

$$(3) \quad \#\{z_1, \dots, z_{2n}, z'_1, \dots, z'_{2n}\} \geq 2n + 1$$

The existence of such an index follows from the condition $(\mathbb{R}^2, 0) \nparallel \text{Aff}(Q')$ as follows. Let w'_k be on the minimal height among the vertices of Q'_α , as measured by the third coordinate. There can be at most one more vertex of Q'_α on the same height, and if such exists it must be adjacent to w'_k . We can assume that w'_{k+1} is strictly higher than w'_k . Consider the plane H_{n+k} through the edge of Q_α , opposite to $[w_k, w_{k+1}]$. The hight function on the $2n$ -gon $H_{n+k} \cap (Q \times \mathbb{R}_{\geq 0})$ is maximized along the segment $H_{n+k} \cap ([w_k, w_{k+1}] \times \mathbb{R}_{\geq 0})$. In particular, H_{n+k} has the desired property: $H_{n+k} \cap Q'_\alpha = \{w'_k\}$.

Using the same argument for the vertices z'_1, \dots, z'_{2n} , one strengthens (3) to the inequality to $\#\{z_1, \dots, z_{2n}, z'_1, \dots, z'_{2n}\} \geq 2n + 2$. We also have the two vertices of $f_*^{-1}(\alpha)$, corresponding to the planes $(\mathbb{R}^2, 0)$ and $\text{Aff}(Q')$. Since they do not belong to $\{z_1, \dots, z_{2n}, z'_1, \dots, z'_{2n}\}$, the last inequality proves (2).

A generic element of $\text{Im}(f_*)$ has rank = 2. Therefore, the inequality (2), Lemma 3.1(b), and Proposition 3.4(a) imply $\#\text{vert}(\Sigma \text{Hom}(R, f)) \geq 2n + 4$.

On the other hand, since $\Sigma f \cong [0, 1]$, (1) implies $\text{Hom}(Q, \Sigma f) \cong \diamond(Q^\circ)$, and this in turn implies $\#\text{vert}(\text{Hom}(Q, \Sigma f)) = 2n + 2$. \square

5. MINKOWSKI SUM COVARIANCE

Before developing a correct version of the fiber equality in Section 6, we investigate the functorial behavior of the Minkowski sum. The following well known fact (e.g., [5, Proposition 2.1]) will be useful:

Lemma 5.1. *For $P, Q \in \mathbf{Pol}$, the polytope $\text{Hom}(P, Q)$ has dimension is $(\dim P + 1) \dim Q$ and the facets $H(v, F) = \{\varphi \in \text{Hom}(P, Q) \mid \varphi(v) \in F\}$, where $v \in \text{vert}(P)$ and $F \in \mathbb{F}(Q)$.*

Lemma 5.2. *Let Q and R be polytopes in a vector space V . Then for any polytope P we have $\text{Hom}(P, Q + R) = \text{Hom}(P, Q) + \text{Hom}(P, R)$.*

Proof. First we reduce the general case to the case when Q and R are full-dimensional and $\mathcal{N}(Q) = \mathcal{N}(R)$.

Without loss of generality, $0 \in Q \cap R$ and $V = \mathbb{R}Q + \mathbb{R}R$. For a real number $t > 0$ consider the polytopes $Q_t = Q + tR$ and $R_t = R + tQ$. We have:

- $\dim(Q_t) = \dim(R_t) = \dim V$,
- $\mathcal{N}(Q_t) = \mathcal{N}(R_t)$ (Lemma 3.1(a)),
- $\lim_{t \rightarrow 0} Q_t = Q$ and $\lim_{t \rightarrow 0} R_t = R$.

By Lemma 3.6(a,b), it is enough to prove Lemma 5.2 for Q_t and R_t with $t > 0$ sufficiently small. This way we have reduced the general case to full-dimensional polytopes with equal normal fans.

By Lemma 3.1(a), $\mathcal{N}(Q) = \mathcal{N}(R) = \mathcal{N}(Q + R)$. By Lemma 5.1, this equality implies that for a vertex $x \in P$ and a pair of corresponding facets $F \subset Q$ and $G \subset R$, the three facets $H(x, F) \subset \text{Hom}(P, Q)$, $H(x, G) \subset \text{Hom}(P, R)$, and $H(x, F + G) \subset \text{Hom}(P, Q + R)$ are parallel, i.e., represent a same 1-cone in the common normal fan (notation as in Lemma 5.1). Using again Lemma 5.1 and Lemma 3.1(a), we have

$$\begin{aligned} \mathcal{N}(\text{Hom}(P, Q)) &= \mathcal{N}(\text{Hom}(P, R)) = \mathcal{N}(\text{Hom}(P, Q + R)) \\ &= \mathcal{N}(\text{Hom}(P, Q) + \text{Hom}(P, R)). \end{aligned}$$

Consequently, it is enough to show that the interiors of corresponding pairs of facets of $\text{Hom}(P, Q + R)$ and $\text{Hom}(P, Q) + \text{Hom}(P, R)$ meet.

Lemma 5.1 implies that the interior points of the facets $H(x, F) \subset \text{Hom}(P, Q)$ and $H(x, G) \subset \text{Hom}(P, R)$ are, respectively, the sets

$$\begin{aligned} \{f \in H(x, F) \mid f(x) \in \text{int}(F), f(\text{vert}(P) \setminus \{x\}) \subset \text{int}(Q)\} \\ \{g \in H(x, G) \mid g(x) \in \text{int}(G), g(\text{vert}(P) \setminus \{x\}) \subset \text{int}(R)\}. \end{aligned}$$

By Lemma 3.1(c), for such f and g the sum $f + g$ is in the interior of the corresponding facet of $\text{Hom}(P, Q) + \text{Hom}(P, R)$. But it is also in the interior of the facet $H(x, F + G) \subset \text{Hom}(P, Q + R)$ by the similar description of the latter. \square

Remark 5.3. In the proof of Lemma 5.2, the initial reduction to polytopes with equal normal fans seems unavoidable. The reason for this is the lack of control of the normal fan of $\text{Hom}(P, Q) + \text{Hom}(P, R)$ for general Q and R . It is this reduction step where limits enter the picture, even if one wants to prove Lemma 5.2 for full-dimensional polytopes. On the other hand, a convex set is the same as a filtered union of polytopes. This, together with the uniqueness of limits, explains why **Comp** is the optimal framework for our functorial approach.

Example 5.4. The contravariant version of Lemma 5.2 is false; i.e., in general, $\text{Hom}(P, R) + \text{Hom}(Q, R) \not\cong \text{Hom}(P + Q, R)$. Consider the following rectangles: $P = \text{conv}((2, 1), (2, -1), (-2, 1), (-2, -1))$, $Q = \text{conv}((1, 2), (1, -2), (-1, 2), (-1, -2))$, $R = [0, 1]$. Since $P + Q = [-3, 3]^2$, by (1) we have $\text{Hom}(P + Q, R) \cong \diamond(\square_2)$. On the other hand, the polar polytopes P° and Q° are central parallelograms in \mathbb{R}^2 , related by a 90° -rotation and with unequal diagonals along the coordinate axes. In particular, $P^\circ + Q^\circ$ is an central octagon in \mathbb{R}^2 . Together with (1) this implies that the bipyramids $\text{Hom}(P, R) \cong \diamond(P^\circ)$ and $\text{Hom}(Q, R) \cong \diamond(Q^\circ)$ are related by a 90° -rotation around the axis $\mathbb{R}(0, 0, 1)$, implying in turn $\text{Hom}(P, R) + \text{Hom}(Q, R) \cong \diamond(P^\circ + Q^\circ)$. But the polytopes $\diamond(P^\circ + Q^\circ)$ and $\diamond(\square_2)$ have different numbers of facets: 16 vs. 8.

Corollary 5.5. (a) *Let $Q_1, \dots, Q_m \in \mathbf{Comp}$ be in a same vector space. Then $\text{Hom}(-, Q_1 + \dots + Q_m) \cong \text{Hom}(-, Q_1) + \dots + \text{Hom}(-, Q_m)$ in $\mathbf{Comp}^{\mathbf{Comp}^{\text{op}}}$, or in $\mathbf{Pol}^{\mathbf{Pol}^{\text{op}}}$ if $Q_1, \dots, Q_m \in \mathbf{Pol}$.*

- (b) Let $\varepsilon, \lambda_1, \dots, \lambda_k > 0$, $0 < t < \varepsilon$, and $i = 1, \dots, k$. Assume $X, X_t, Y_i, Y_{it} \in \mathbf{Comp}$, $\text{Aff}(X) = \text{Aff}(X_t)$, and $Y, Y_{it} \subset V \in \mathbf{Vect}$. Assume $\lim_{t \rightarrow 0} X_t = X$ and $\lim_{t \rightarrow 0} Y_{it} = Y_i$. Then $\lim_{t \rightarrow 0} \text{Hom}(X_t, \lambda_1 Y_{t1} + \dots + \lambda_k Y_{tk}) = \lambda_1 \text{Hom}(X, Y_1) + \dots + \lambda_k \text{Hom}(X, Y_k)$ in $\text{Hom}(\text{Aff}(X), V)$.

Proof. (a) By Lemma 5.2, the tautological embedding

$$\text{Hom}(P, Q_1) + \dots + \text{Hom}(P, Q_m) \hookrightarrow \text{Hom}(P, Q_1 + \dots + Q_m)$$

is surjective. But it is also natural in P .

(b) In view of Lemmas 3.6(a,b) and 5.2, one only needs to represent the convex sets as limits of polytopes and use $\text{Hom}(X, \lambda Y) = \lambda \text{Hom}(X, Y)$. \square

6. AFFINE-COMPACT KERNEL

By analogy with the functor $\text{Hom}_{\mathbf{Vect}}(-, \ker \alpha) : \mathbf{Vect} \rightarrow \mathbf{Vect}$, for a map $f : X \rightarrow Y$ in \mathbf{Comp} we introduce the following contravariant functor:

$$\begin{aligned} \text{Hom}(-, \ker_{\text{Aff}}(f)) &: \mathbf{Comp} \rightarrow \mathbf{Sets}, \\ \text{Hom}(Z, \ker_{\text{Aff}}(f)) &= \{g : Z \rightarrow X \mid g \text{ affine and } f(g(Z)) \text{ a singleton}\}, \\ \text{Hom}(h, \ker_{\text{Aff}}(f))(g) &= gh \quad \text{for } h : Z' \rightarrow Z \text{ in } \mathbf{Comp}. \end{aligned}$$

We will need the following *pull-back diagram* in \mathbf{Sets} , which also introduces the map $\text{Hom}(Z, f)^{\text{ev}}$:

$$(4) \quad \begin{array}{ccc} \text{Hom}(Z, \ker_{\text{Aff}}(f)) & \hookrightarrow & \text{Hom}(Z, X) \\ \text{Hom}(Z, f)^{\text{ev}} \downarrow & & \downarrow \text{Hom}(Z, f) \\ Y & \xrightarrow{\text{const.}} & \text{Hom}(Z, Y) \end{array}$$

where (i) to every point $y \in Y$ the bottom map assigns the constant map $Z \rightarrow Y$ with value y , and (ii) $\text{Hom}(Z, f)^{\text{ev}}$ is the evaluation map $g \mapsto (fg)(Z)$.

Proposition 6.1. *In the notation introduced above,*

- (a) $\text{Hom}(-, \ker_{\text{Aff}}(f))$ is in $\mathbf{Comp}^{\mathbf{Comp}^{\text{op}}}$, or in $\mathbf{Pol}^{\mathbf{Pol}^{\text{op}}}$ if f is in \mathbf{Pol} .
- (b) $\text{Hom}(-, \ker_{\text{Aff}}(f))$ is not a representable functor unless $f(X)$ is a singleton.
- (c) In $\mathbf{Comp}^{\mathbf{Comp}^{\text{op}}}$, or in $\mathbf{Pol}^{\mathbf{Pol}^{\text{op}}}$ if f is in \mathbf{Pol} , we have

$$\text{Hom}(-, \ker_{\text{Aff}}(f)) = \varinjlim (\text{Hom}(-, f^{-1}(y)) \mid y \in Y).$$

- (d) $\dim(\text{Hom}(Z, \ker_{\text{Aff}}(f))) = (\dim Z + 1) \text{codim } f + \dim(f(X))$,
- (e) $\text{Im}(\text{Hom}(Z, \ker_{\text{Aff}}(f))) = f(X)$.

Proof. For (a), since the assignment $h \mapsto gh$ is an affine function of g , we only need to show $\text{Hom}(Z, \ker_{\text{Aff}}(f)) \in \mathbf{Comp}$, with $\text{Hom}(Z, \ker_{\text{Aff}}(f)) \in \mathbf{Pol}$ for $X, Y, Z \in \mathbf{Pol}$. But because the limits in \mathbf{Sets} , \mathbf{Comp} , \mathbf{Pol} agree, this claim follows from the pull-back diagram (4), in which the right and bottom arrows represent affine maps.

(d) This follows from the dimension formula in Lemma 5.1 and the equality $\dim(f^{-1}(y)) = \text{codim } f$ for generic $y \in f(X)$.

(e) The inclusion $\text{Im}(\text{Hom}(Z, \ker_{\text{Aff}}(f))) \subset f(X)$ is obvious and the opposite inclusion follows by considering the constant maps $Z \rightarrow X$.

(b) If there is an ‘affine-compact kernel object’ $\ker_{\text{Aff}}(f) \in \mathbf{Comp}$ then we have $\dim(\text{Hom}(Z, \ker_{\text{Aff}}(f))) = (\dim Z + 1) \dim(\ker_{\text{Aff}}(f))$ (Lemma 5.1). But by (d), $\dim(\text{Hom}(Z, \ker_{\text{Aff}}(f))) = (\dim Z + 1) \text{codim } f + \dim f(X)$ for *every* Z . This is a contradiction unless $\dim(f(X)) = 0$.

(c) Every functor in $\mathbf{Sets}^{\mathbf{Comp}^{\text{op}}}$ is a colimit of representable functors. This is true for any functor from any category to \mathbf{Sets} , covariant or contravariant – a consequence of *Yoneda Lemma* [10, Ch.2], whose enriched version over symmetric monoidal categories is worked out in [9, Ch.2]. In the case of $\text{Hom}(-, \ker_{\text{Aff}}(f))$ checking the given colimit equality is straightforward. It also follows from the standard recipe for such colimit representations, given in [10, Ch.3, §7] (where the details are worked out for the dual/covariant case). That recipe produces a much larger non-discrete diagram. However, in our case the connected components of the resulting diagram have colimits exactly the representable functors in (c). (Beware of the typos in the proof of [10, Theorem III.7.1]: J^D in the definition of M as well as J in the diagram (1) are supposed to be J^{op} .) \square

Next, for a map $f : X \rightarrow Y$ in \mathbf{Comp} , we introduce the following functor

$$\Sigma \text{Hom}(-, f)^{\text{ev}} \in \mathbf{Comp}^{\mathbf{Comp}^{\text{op}}}.$$

To $Z \in \mathbf{Comp}$ it assigns $\Sigma \text{Hom}(Z, f)^{\text{ev}}$. For an affine map $h : Z' \rightarrow Z$ we first work out the polytopal case, i.e., when $X, Y, Z \in \mathbf{Pol}$. By Proposition 6.1(f), $\text{Im}(\text{Hom}(Z, f)^{\text{ev}}) = \text{Im}(\text{Hom}(Z', f)^{\text{ev}}) = f(X)$. Let $\sigma_1, \dots, \sigma_n$ be the maximal cells of a subdivision of $f(X)$, which subdivides the images of faces of both polytopes $\text{Hom}(Z, \ker_{\text{Aff}}(f))$ and $\text{Hom}(Z', \ker_{\text{Aff}}(f))$. Then, using Proposition 3.4(a), we can define the map $\text{Hom}(h, f)^{\text{ev}} : \Sigma \text{Hom}(Z, f)^{\text{ev}} \rightarrow \Sigma \text{Hom}(Z', f)^{\text{ev}}$ as follows:

$$\begin{aligned} \sum_{i=1}^n \frac{\text{vol}(\sigma_i)}{\text{vol}(f(X))} \cdot (\text{Hom}(Z, f)^{\text{ev}})^{-1}(x_j) & \xlongequal{\quad} \Sigma \text{Hom}(Z, f)^{\text{ev}} \\ & \parallel \\ \sum_{i=1}^n \frac{\text{vol}(\sigma_i)}{\text{vol}(f(X))} \cdot \text{Hom}(Z, f^{-1}(x_j)) & \xrightarrow{- \circ h} \sum_{i=1}^n \frac{\text{vol}(\sigma_i)}{\text{vol}(f(X))} \cdot \text{Hom}(Z', f^{-1}(x_j)) \\ & \parallel \\ \Sigma \text{Hom}(Z', f)^{\text{ev}} & \xlongequal{\quad} \sum_{i=1}^n \frac{\text{vol}(\sigma_i)}{\text{vol}(f(X))} \cdot (\text{Hom}(Z', f)^{\text{ev}})^{-1}(x_j) \end{aligned}$$

Checking that we get a functor in $\mathbf{Pol}^{\mathbf{Pol}^{\text{op}}}$ is straightforward, with a similar use of Proposition 3.4(a).

Now assume Z and $f : X \rightarrow Y$ are in \mathbf{Comp} , $Z = \lim_{t \rightarrow 0} Z_t$, and $X = \lim_{t \rightarrow 0} X_t$, where $Z_t \subset Z$ and $X_t \subset X$ are in \mathbf{Pol} , satisfying the condition $\dim Z_t = \dim Z$ and $\dim(X_t) = \dim X$ for all $0 < t < \varepsilon$. Put $f_t = f|_{X_t} : X_t \rightarrow f(X_t)$. Lemma 3.6(b) and the pull-back diagrams for Z_t and f_t , similar to (4), imply the

convergence $\lim_{t \rightarrow 0} \text{Hom}(Z_t, \ker_{\text{Aff}}(f_t)) = \text{Hom}(Z, \ker_{\text{Aff}}(f))$ in the ambient space $\text{Hom}(\text{Aff}(Z), \text{Aff}(X))$. Then Lemma 3.6(c) implies

$$(5) \quad \lim_{t \rightarrow 0} \Sigma \text{Hom}(Z_t, f_t)^{\text{ev}} = \Sigma \text{Hom}(Z, f)^{\text{ev}}.$$

Let $h : Z' \rightarrow Z$ be in **Comp** and $Z' = \lim_{t \rightarrow 0} Z'_t$ with $\dim(Z'_t) = \dim Z$ for all t . We can additionally assume $h(Z'_t) \subset Z_t$ for all t . Denote $h_t := h|_{Z'_t} : Z'_t \rightarrow Z_t$. The definition of our functor in the polytopal case above ensures the compatibility $\text{Hom}(h_t, f_t)^{\text{ev}} = \text{Hom}(h_s, f_s)^{\text{ev}}$ on $\text{Hom}(Z_t, f)^{\text{ev}} \cap \text{Hom}(Z_s, f)^{\text{ev}}$. In particular, the limit equality (5) gives rise to a functorial map $\Sigma \text{Hom}(Z, f)^{\text{ev}} \rightarrow \Sigma \text{Hom}(Z', f)^{\text{ev}}$.

We are ready to state the following affine substitute for Σ -covariance.

Theorem 6.2. *If f is in **Comp** then $\Sigma \text{Hom}(-, f)^{\text{ev}} \cong \text{Hom}(-, \Sigma f)$ in **Comp**^{Comp^{op}}, or in **Pol**^{Pol^{op}} if f is in **Pol**.*

Proof. First we consider the case, when Z and $f : X \rightarrow Y$ are in **Pol**. By Proposition 6.1(e), $\text{Im}(\text{Hom}(Z, f)^{\text{ev}}) = f(X)$. Let $\sigma_1, \dots, \sigma_n \subset Y$ be the maximal cells of a subdivision of $f(X)$, which subdivides the images of faces of $\text{Hom}(X, Y)$ as well as the of faces of X . We have

$$\begin{aligned} \Sigma \text{Hom}(Z, f)^{\text{ev}} &= \sum_{j=1}^m \frac{\text{vol}(\sigma_j)}{\text{vol}(f(X))} \cdot \text{Hom}(Z, f^{\text{ev}})^{-1}(x_j) = \\ &= \sum_{j=1}^m \frac{\text{vol}(\sigma_j)}{\text{vol}(f(X))} \cdot \text{Hom}(Z, f^{-1}(x_j)) \cong \text{Hom} \left(Z, \sum_{j=1}^m \frac{\text{vol}(\sigma_j)}{\text{vol}(f(X))} \cdot f^{-1}(x_j) \right) = \text{Hom}(Z, \Sigma f), \end{aligned}$$

where the first and last equalities follow from Proposition 3.4(a) and the middle isomorphism is provided by Corollary 5.5(a).

The general case, when Z and f are in **Comp**, can be derived from the polytopal case along the lines the functor $\Sigma \text{Hom}(-, f)^{\text{ev}} \in \mathbf{Comp}^{\mathbf{Comp}^{\text{op}}}$ was constructed in two steps, first considering the polytopal case. \square

7. AFFINE-COMPACT COKERNEL

The following definition is modeled after the isomorphisms $\text{Hom}_{\mathbf{Vect}}(\text{coker } \alpha, -) \cong \ker \text{Hom}_{\mathbf{Vect}}(\alpha, -)$ and $\text{Hom}_{\mathbf{Vect}}(-, \text{coker } \alpha) \cong \text{coker } \text{Hom}_{\mathbf{Vect}}(-, \alpha)$.

Definition 7.1. For a map $f : X \rightarrow Y$ in **Comp**, we have the *object*

$$\text{coker}_{\text{Aff}} * f = \lim_{\substack{\longrightarrow \\ f(x)}} \left(X \begin{array}{c} \xrightarrow{f} \\ \xleftarrow{f(x)} \end{array} Y \right) \in \mathbf{Comp},$$

where the lower arrow in the diagram is a constant map of the form $f(X) = f(x)$ for some $x \in X$, and the *functor* $\text{Hom}(-, \text{coker}_{\text{Aff}}^* f) \in \mathbf{Sets}^{\mathbf{Comp}^{\text{op}}}$, defined by

$$\begin{aligned} \text{Hom}(Z, \text{coker}_{\text{Aff}}^* f) &= \text{Hom}(Z, Y) / (g_1 \sim g_2 \text{ iff } \rho g_1 = \rho g_2 \text{ for any } \rho : Y \rightarrow Y' \\ &\quad \text{in } \mathbf{Comp} \text{ with } (\rho f)(X) \text{ a singleton}), \\ \text{Hom}(h, \text{coker}_{\text{Aff}}^* f)([g]) &= [gh] \quad \text{for } h : Z' \rightarrow Z \text{ and } g : Z \rightarrow Y \text{ in } \mathbf{Comp}. \end{aligned}$$

First we observe that $\text{coker}_{\text{Aff}*} f$ is independent of the choice of $x \in X$ and it can be identified with $\pi(Y)$, where $\pi : \text{Aff}(Y) \rightarrow \text{Aff}(Y)$ is an affine map with $\pi^{-1}(\pi(f(x))) = \text{Aff}(f(X))$; i.e., π is a linear projection of Y along $\text{Aff}(X)$.

The functor $\text{Hom}(\text{coker}_{\text{Aff}*} f, -) \in \mathbf{Sets}^{\mathbf{Comp}}$ directly identifies as follows:

$$\begin{aligned} \text{Hom}(\text{coker}_{\text{Aff}*} f, Z) &= \{g : Y \rightarrow Z \mid g \text{ affine and } g(f(X)) \text{ a singleton}\}, \\ \text{Hom}(\text{coker}_{\text{Aff}*} f, h)(g) &= hg \quad \text{for } g : Y \rightarrow Z \text{ and } h : Z \rightarrow Z' \text{ in } \mathbf{Comp}. \end{aligned}$$

Using barycentric coordinates, one easily verifies that

- $\text{Hom}(\text{coker}_{\text{Aff}*} f, -)$ is in $\mathbf{Comp}^{\mathbf{Comp}}$, or in $\mathbf{Pol}^{\mathbf{Pol}}$ if f is in \mathbf{Pol} ,
- $\text{Hom}(-, \text{coker}_{\text{Aff}*}^* f)$ is in $\mathbf{Comp}^{\mathbf{Comp}^{\text{op}}}$, or in $\mathbf{Pol}^{\mathbf{Pol}^{\text{op}}}$ if f is in \mathbf{Pol} .

The next proposition clarifies the relationship between the two cokernels. First we observe that any map $\varphi : S \rightarrow T$ in \mathbf{Comp} gives rise to the functor:

$$\begin{aligned} \text{Im}(\text{Hom}(-, \varphi)) &\in \mathbf{Comp}^{\mathbf{Comp}^{\text{op}}}, \\ \text{Im}(\text{Hom}(Z, \varphi)) &= \text{Im}(\text{Hom}(Z, S) \xrightarrow{\text{Hom}(Z, \varphi)} \text{Hom}(Z, T)), \\ \text{Im}(\text{Hom}(h, \varphi))(g) &\mapsto gh \quad \text{for } h : Z' \rightarrow Z \text{ in } \mathbf{Comp} \text{ and } g \in \text{Im}(\text{Hom}(Z, \varphi)). \end{aligned}$$

Proposition 7.2. *Let $f : X \rightarrow Y$ be in \mathbf{Comp} and $\pi : Y \rightarrow \text{coker}_{\text{Aff}*} f$ be the canonical map.*

- (a) $\text{Hom}(-, \text{coker}_{\text{Aff}*}^* f) \cong \text{Im}(\text{Hom}(-, \pi))$ in $\mathbf{Comp}^{\mathbf{Comp}^{\text{op}}}$
- (b) $\text{Hom}(-, \text{coker}_{\text{Aff}*}^* f)$ is representable if and only if π has an affine section, in which case $\text{coker}_{\text{Aff}*}^* f = \text{coker}_{\text{Aff}*} f$.

Proof. Part (a) is straightforward because $[g_1] = [g_2]$ in $\text{Hom}(Z, \text{coker}_{\text{Aff}*}^* f)$ if and only if $\pi g_1 = \pi g_2$.

(b) If π has a section $\sigma : \text{coker}_{\text{Aff}*} f \rightarrow Y$ then $\text{Hom}(-, \sigma)$ is a right inverse of $\text{Hom}(-, \pi)$. In particular, $\text{Hom}(Z, \pi)$ is surjective for every Z .

Conversely, if $\text{Hom}(Z, \pi)$ is surjective for every Z then a preimage of $1_{\text{coker}_{\text{Aff}*} f} \in \text{Hom}(\text{coker}_{\text{Aff}*} f, \text{coker}_{\text{Aff}*} f)$ in $\text{Hom}(Y, \text{coker}_{\text{Aff}*} f)$ is a section of π . \square

Remark 7.3. The cokernel does not correlate well with the fiber construction: usually $\text{Hom}(\text{coker}_{\text{Aff}*} f, Z) \not\cong \Sigma \text{Hom}(f, Z)$. For example, if $X = Y$ and $f = 1_X$ then $\text{Hom}(\text{coker}_{\text{Aff}*} f, Z) \cong Z$ and $\Sigma \text{Hom}(f, Z)$ is a point.

8. SANDWICHING AND COMPLEMENTING

Let $Y \subset X$ be in \mathbf{Comp} and $\dim Y = \dim X = d$. For a d -simplex Δ , the *space of sandwiched simplices* $\Delta_{Y,X}$, introduced in [11], is the subset $\{g : \Delta \rightarrow X \mid Y \subset g(\Delta)\} \subset \text{Hom}(\Delta, X)$. This is a complicated topological space and [11] employs Morse theory in its analysis.

One way to generalize the construction to arbitrary $X, Y, Z \in \mathbf{Comp}$ is to consider the subset $Z_{Y,X} = \{g : Z \rightarrow X \mid Y \subset g(Z)\} \subset \text{Hom}(Z, X)$. But $Z_{Y,X}$ is functorial neither in Z nor in Y . This can be partially fixed using of the *category of factorizations* in \mathbf{Comp} , in the sense of [2]. (It is different from the *category of*

arrows [10, Ch.2]). The objects of **FComp** are the maps in **Comp** and a morphism from $f' : Y' \rightarrow X'$ to $f : Y \rightarrow X$ is a commutative square

$$\begin{array}{ccc} Y & \xrightarrow{f} & X \\ \varphi \uparrow & & \downarrow \vartheta \\ Y' & \xrightarrow{f'} & X' \end{array}$$

In this notation, every $Z \in \mathbf{Comp}$ gives rise to the *sandwiching* functor:

$$\begin{aligned} \Xi[Z] : \mathbf{FComp} &\rightarrow \mathbf{Sets}, \\ \Xi[Z](f) &= \{g \in \text{Hom}(Z, X) \mid f(Y) \subset g(Z) \subset X\}, \\ \Xi[Z](\varphi, \vartheta)(-) &= \vartheta \cdot - \end{aligned}$$

Observe, $\Xi[Z]$ is affine-compact covariant in X and affine-compact contravariant in Y . Ξ is also affine-compact contravariant in Z , but only for surjective maps.

There is an associated and in a sense complementary assignment, more amenable to topological control and functorial in Z and Y (and X for injective maps):

Definition 8.1. For a $X \in \mathbf{Comp}$ we have the functor:

$$\begin{aligned} \text{Hom}(-, X \setminus -) : \mathbf{Comp}^{\text{op}} \times (\mathbf{Comp} \downarrow X)^{\text{op}} &\rightarrow \mathbf{Sets}, \\ \text{Hom}(Z, X \setminus \varphi) &= \overline{\{g : Z \rightarrow X \mid g \text{ affine and } g(Z) \cap \text{int}(\text{Im } \varphi) = \emptyset\}}, \\ \text{Hom}(h, X \setminus t)(g) &= gh, \\ \text{for } h : Z' \rightarrow Z \text{ and } & \begin{array}{ccc} Y' & \xrightarrow{t} & Y \\ & \searrow \varphi' & \swarrow \varphi \\ & X & \end{array} \text{ a commutative triangle,} \end{aligned}$$

where \downarrow is for the comma category and the overline is for the topological closure.

An informal link to cokernel objects of the two functors is that sandwiching corresponds to making $\varphi(Y)$ a necessary target inside X , similar to 0 in the abelian situation, and complementing corresponds to making $\text{int}(\varphi(Y))$ an impossible target, a fusion of the topological quotient and affine maps.

Below, using the notation in Definition 8.1, we list several easily checked facts:

- $\text{Hom}(-, X \setminus -)$ is an affine-compact bifunctor and $\text{Hom}(Z, X \setminus \varphi)$ is never convex;
- If $\dim(\text{Im } \varphi) = \dim X$ then $\Xi[P](\varphi) \cap \text{Hom}(X, \setminus \varphi) = \emptyset$;
- $\text{Hom}(Z, X \setminus 1_X) = \text{Hom}(Z, \partial X)$, where ∂X and $\text{Hom}(Z, \partial X)$ are viewed as polytopal complexes in the sense of [1];
- If $\dim(\text{Im } \varphi) = \dim X$ then taking the topological closure is superfluous;
- If $\dim(\text{Im } \varphi) + \dim Z < \dim X$ then $\text{Hom}(Z, X \setminus \varphi) = \text{Hom}(Z, X)$;
- $\text{Hom}(Z, X \setminus \varphi) = \bigcup \text{Hom}(Z, X')$, where the union is taken inside $\text{Hom}(Z, X)$ over all $X' \subset X$, $X' \in \mathbf{Comp}$, admitting affine surjective maps $\alpha : \text{Aff}(X) \rightarrow \mathbb{R}$, such that $\alpha(X') \subset \mathbb{R}_{\geq 0}$ and $\alpha(\text{Im } \varphi) \subset \mathbb{R}_{\leq 0}$.

- The following colimit equality holds in $\mathbf{Sets}^{\mathbf{Comp}^{\text{op}}}$

$$\begin{aligned} \text{Hom}(-, X \setminus \varphi) = \varinjlim \left(\text{Hom}(-, X') : X' \text{ as above, together with the natural} \right. \\ \left. \text{transformations } \text{Hom}(-, X') \xrightarrow{\bullet} \text{Hom}(-, X'') \right. \\ \left. \text{whenever } X' \subset X'' \right). \end{aligned}$$

Call a set in a vector space *polytopal* if it is covered by finitely many polytopes. For a polytope P , let \tilde{P} denote the diagram of the codimension 1 and 2 faces of P and the embeddings between them.

Theorem 8.2. *Let X, Y, Z and $\varphi : Y \rightarrow X$ be in \mathbf{Comp} .*

- (a) *If X, Z and φ are in \mathbf{Pol} then $\text{Hom}(Z, X \setminus \varphi)$ is semialgebraic and so is $\Xi[Z](\varphi)$ when $\dim(\text{Im } \varphi) = \dim X = \dim Z$, in general neither of them polytopal.*
- (b) *If $\dim(\text{Im } \varphi) = \dim X$ then ∂X is a strong deformation retract of $\text{Hom}(Z, X \setminus \varphi)$. In all other cases $\text{Hom}(Z, X \setminus \varphi)$ is contractible.*
- (c) (Complementarity) *Assume P is a polytope and $\dim(\text{Im } \varphi) = \dim X = \dim P$. There exists a natural injective map of sets*

$$\rho : \Xi[P](\varphi) \amalg \text{Hom}(P, X \setminus \varphi) \rightarrow \varprojlim \text{Hom}(\tilde{P}, X \setminus \varphi),$$

which is bijective if P is simple, except an n -gon with $n \geq 4$. If P is simplicial, not a simplex, and $Y \subset \text{int}(X)$ then ρ is not bijective.

(The limit in (c) is taken in \mathbf{Sets} .)

Proof. (a) Consider the evaluation map $\text{ev} : \text{Hom}(Z, X) \times Z \rightarrow X$. It is not affine, but *bi-affine*: upon fixing one component the map is affine in the other. Let $\pi : \text{Hom}(Z, X) \times Z \rightarrow \text{Hom}(Z, X)$ be the projection map. Then

$$\text{Hom}(Z, X \setminus \varphi) = \overline{\pi((\text{Hom}(Z, X) \times Z) \setminus \text{ev}^{-1}(\text{int}(\text{Im } \varphi)))}.$$

Since ev is a degree 2 polynomial map basic properties of semialgebraic sets (e.g., the Tarski-Seidenberg theorem) [4] guarantee that $\text{Hom}(Z, X \setminus \varphi)$ is semialgebraic.

As for the set $\Xi[Z](\varphi)$, the inclusion $\varphi(Y) \subset g(Z)$ is equivalent to the condition that the isomorphic images of the facets of Z have the vertices of $\varphi(Y)$ on the corresponding non-negative sides, leading to a system of nonstrict determinantal inequalities where the matrix entries are changed to the linear forms, defining g . The nondegeneracy condition of g adds two more determinantal strict inequalities.

Consider two concentric squares $\square' \subset \square'' \subset \mathbb{R}^2$, with edges parallel to the coordinate axis and \square'' sufficiently larger than \square' . Let Δ be a triangle. Then neither $\Xi[\Delta](\iota)$ nor $\text{Hom}([0, 1], \square'' \setminus \iota)$ is polytopal for $\iota : \square' \rightarrow \square''$ the identity embedding. Without delving into the planar geometry details, we only mention that this follows from the strict convexity of the function, assigning the x -coordinate of a point p in the upper edge of \square'' the y -coordinate of the point q on the right edge of \square'' , such that the upper right corner of \square' is in the segment $[p, q]$.

(b) Assume $\dim(\text{Im } \varphi) = \dim X$. We think of $X \setminus \text{int}(\text{Im } \varphi)$ as a subset of $\text{Hom}(Z, X \setminus \varphi)$ via identifying every point $x \in X \setminus \text{int}(\text{Im } \varphi)$ with the constant map $g : Z \rightarrow X$, $g(Z) = x$. Pick a point $z \in Z$. The homotopy

$$\begin{aligned} H_t : \text{Hom}(Z, X \setminus \varphi) &\rightarrow \text{Hom}(Z, X \setminus \varphi), \quad t \in [0, 1], \\ g &\mapsto (\text{homothety of } \text{Aff}(X) \text{ with coefficient } t \text{ and} \\ &\quad \text{centered at } g(z)) \circ g \end{aligned}$$

makes $X \setminus \text{int}(\text{Im } \varphi)$ a strong deformation retract of $\text{Hom}(Z, X \setminus \varphi)$. But ∂X is a strong deformation retract of $X \setminus \text{int}(\text{Im } \varphi)$ via the projection onto the boundary from a point $y \in \text{int}(\text{Im } \varphi)$. By concatenation we get the desired deformation retraction.

If $\dim Y < \dim X$ then the homotopy above makes X itself a deformation retract of $\text{Hom}(Z, X \setminus \varphi)$, and X is contractible.

(c) The limit can be thought of as the set of face-wise affine maps $\gamma : \partial P \rightarrow X$, satisfying $\gamma|_F \in \text{Hom}(F, X \setminus \varphi)$ for every $F \in \mathbb{F}(P)$. The equalities $\dim(\text{Im } \varphi) = \dim X = \dim P$ imply that if a map γ as above extends to an affine map $g : P \rightarrow X$ then either $\varphi(Y) \subset g(P)$ or $g(P) \cap \text{int}(\varphi(Y)) = \emptyset$. Therefore, the assignment $g \mapsto g|_{\partial P}$ gives rise to an injective map with the mentioned source and target. It is injective because two affine maps from P coincide if they agree on ∂P .

Let P be simple and $\gamma : \partial P \rightarrow X$ be in the limit. We want to show that γ extends to a map $g : P \rightarrow X$. Pick a vertex $v \in P$. Because v is simple, there exists a unique affine map $g : P \rightarrow \text{Aff}(X)$, which agrees with γ on the facets $F \subset P$ with $v \in F$. We claim that γ and g agree on *all* facets of P . Observe that if γ and g agree on all facets containing some vertex except possibly one then, because P is simple, the two maps also agree on the remaining facet. Now the claim is proved by bringing in one by one the facets of P for which the equality $g = \gamma$ has been verified, starting with the initial set of facets through v : in this process, until all facets have been incorporated, there is always a vertex, incident with exactly one new facet.

Finally, assume P is simple, not a simplex. There is a vertex $v \in P$, such that P is not a pyramid with apex at v . Let $\Delta_1, \dots, \Delta_n \subset P$ be the codimension 2 faces in the *link* of v in the simplicial complex ∂P . There is a point $w \in \text{Aff}(P)$ such that the facets of P , not containing v , and the simplices $\text{conv}(w, \Delta_i)$ do not form the boundary of a polytope (always assumed to be convex). Denote by Π the union of all these simplices. Because P is simplicial, there is a map $\gamma' : \partial P \rightarrow \Pi$, which restricts to an affine bijection on each simplex. We can even achieve γ' to be injective, but this is not necessary. Since $\varphi(Y) \subset \text{int}(X)$, there is an affine injective map $\gamma'' : \text{conv}(\Pi) \rightarrow X \setminus \text{int}(Y)$. Then $\gamma''\gamma'$ is in the limit in (c) but not in $\text{Im } \rho$. \square

Remark 8.3. (*Extended complementarity*) The class of polytopes, for which the map ρ in Theorem 8.2(c) is bijective, is considerably larger than the class of simple polytope. For instance, if the nonsimple vertices of P are isolated in a certain sense than the same argument can apply. As an explicit example, the map ρ is bijective for the 3-polytope $\text{Hom}(P_{2n+1}, [0, 1])$, where P_{2n+1} is the regular $(2n + 1)$ -gon with $n \geq 2$: the hom-polyope has two antipodal non-simple vertices, which are separated by the zig-zagging equator through the other $4n$ vertices – all simple. One can call

a polytope P *affine-rigid* if every face-wise affine map from ∂P to a vector space extends to an affine map from P . A classification of the affine-rigid polytopes is an interesting problem. The isometric counterpart is the much studied *rigidity property* in metric geometry, going back to the *Cauchy Theorem* that all 3-polytopes are rigid.

Question 8.4. Is there a tractable complete, co-complete, self-enriched symmetric monoidal category, extending the structure from **Pol**, and serving as the target for $\Xi[Z]$ and $\text{Hom}(-, X \setminus -)$? For instance, the category of compact (not necessarily convex) semialgebraic sets and affine maps does not seem sufficient. The conical example in [13, Theorem 3.15] hints at the existence of compact convex semialgebraic sets Z and X with $\text{Hom}(Z, X)$ not semialgebraic. Is it true that if X and Z are semialgebraic then so is the subset

$$X \otimes Y := \bigcup_{P, Q \in \mathbf{Pol} \ P \subset X \ Q \subset Y} P \otimes Q \subset \text{Aff}(X) \otimes \text{Aff}(Y)?$$

(The tensor product on the right is the one of affine spaces.)

The existence of a category \mathcal{M} as above would make the functor $\text{Hom}(-, X \setminus \varphi)$ representable whenever $\overline{\text{Im } \varphi}$ is a full-dimensional topological manifold with boundary: $X \setminus \varphi = X \setminus \text{int}(\text{Im } \varphi) \in \mathcal{M}$.

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